THE HYPERRADICAL AND THE HOPKINS–LEVITZKI THEOREM FOR MODULAR LATTICES

FERNANDO GUZMÁN

Abstract. Many arguments in the Theory of Rings and Modules are, on close inspection, purely Lattice theoretic arguments. Călagăreanu has a long repertoire of such results in his book. The Hopkins-Levitzki Theorem is interesting from this point of view, because a special case of it lends to an obvious lattice theory approach, but the rest is a little more subtle. Albu and Smith have obtained some sufficient conditions for the question of when Artinian implies Noetherian. Here we present a new approach, using the concept of Hyperradical; we obtain necessary and sufficient conditions.

1. Introduction

A ring is left (resp. right) Artinian, if its lattice of left (resp. right) ideals satisfies the descending chain condition. It is left (resp. right) Noetherian, if its lattice of left (resp. right) ideals satisfies the ascending chain condition. A classical theorem connecting these concepts is the Hopkins-Levitzki Theorem [6, 8] (HLT for short). It states that every left (resp. right) Artinian ring is left (resp. right) Noetherian. The statement of this theorem is lattice theoretic and it is only natural to ask if there is lattice theoretic proof of it, i.e. if it can be extended to lattices, and under what assumptions. The goal of this paper is to answer that question.

The standard proof of the of the Hopkins-Levitzki Theorem found in algebra textbooks like [7, 5] has two components. The first one considers the special case when the Jacobson radical, defined as the intersection of all maximal ideals, is trivial. This component of the HLT readily extends to lattices; see Proposition 9. In the second component, the ring operations play an essential role, via the nilpotency of the Jacobson radical. So, our question reduces to how this part of the proof can be extended to lattices. Albu and Smith [1, 2, 3] have obtained some results related to our question. In [1] the lattice is assumed to be modular and upper continuous, a condition weaker than algebraic; but a rather technical additional hypothesis, condition (λ), is needed. In

[2], the hypothesis of upper continuity is further weakened to condition $\mathcal{E}$, which ensures that the lattice L has a good supply of essential elements, and condition $\mathcal{BL}$ which places a bound on the composition length of some subintervals. All three conditions, (λ), $\mathcal{E}$, and $\mathcal{BL}$ are local and existential.

Since the lattice of left (right) ideals of a ring is modular and algebraic, it makes sense to being in the lattice case with these two assumptions. It can be seen that modularity alone will not do, by considering $(\mathbb{N}, |)$, the lattice of natural numbers under divisibility. This lattice is Artinian but not Noetherian. Even though this lattice is distributive, hence modular, it is not algebraic. Thus the question arises if every Artinian modular algebraic lattice is Noetherian. The answer is no as we will show in Example 5.

The concept leading to the solution of our problem is the hyperradical, which is a global construction. As we will show, being hyperradical free is not only a sufficient condition, but also necessary for a modular Artinian lattice to be Noetherian. There is no need to assume the lattice to be algebraic, or even upper continuous.

It should be noted that for the lattice of left ideals of a ring, the hyperradical free condition is not vacuous, as will be shown in Example 13.

2. The Radical, Modularity and Chain Conditions

In Ring Theory, there are a number of different *radical* constructions. Some of them, but not all, can be expressed as the intersection of maximal objects in some lattice. One of the best known examples is the Jacobson radical of a ring R, $J(R)$, which is equal to the intersection of all maximal two-sided ideals. It is also equal to the intersection of all maximal left (right) ideals. The extension of this definition to a complete lattice is immediate.

Definition 1. In a complete lattice L, the *radical* of L, $\mathrm{r}(L)$ is the meet of all coatoms of L. If A is a (universal) algebra, we denote by $\mathrm{r}(A)$ the radical of the lattice $\mathrm{Sub}(A)$ of subalgebras of A.

So, the Jacobson radical $J(R)$ is the radical in the lattice of two-sided ideals, as well as the radical in the lattice of left ideals, and in the lattice of right ideals. In other words, it is the radical of R, when viewed as a left R-module.

It is easy to check that in any lattice L, for any $x, y, z \in L$

$$x \leq z \Rightarrow x \vee (y \wedge z) \leq (x \vee y) \wedge z$$

Definition 2. A lattice L is *modular* if it satisfies

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$$

for any $x, y, z \in L$.

If M is a left-module over a ring R, the lattice $\mathrm{Sub}(M)$ of submodules of M is a modular lattice. In particular, the lattice of left-ideals of R is modular.

Definition 3. A lattice L is said to be *Noetherian* or to satisfy the *ascending chain condition*, ACC for short, if it contains no infinite ascending chain $x_0 < x_1 < x_2 < \cdots$
Dually, L is said to be *Artinian* or to satisfy the *descending chain condition*, DCC for short, if it contains no infinite descending chain $x_0 > x_1 > x_2 > \cdots$

A *left-Noetherian Ring* is a ring R such that the lattice of left ideals has the ACC. A *left-Artinian Ring* is a ring R such that the lattice of left ideals has the DCC. We can rephrase the HLT as follows: Let R be a ring and L the lattice of left ideals of R. If L if Artinian, then L is Noetherian.

Not every complete Artinian lattice is Noetherian, as illustrated by the lattice $(\mathbb{N}, |)$. Even though this lattice is distributive, hence modular, it is not algebraic. Since the lattice L of left ideals of a ring R is modular and algebraic, we ask the following question:

Question 4. *Is every Artinian modular algebraic lattice, Noetherian?*

The next example shows that the answer is **no**, and we need to modify the hypotheses.

Example 5. Let L be the lattice of subgroups of Z_{p^∞}. L is a chain isomorphic to $\langle \mathbb{N} \cup \{\infty\}, \leq \rangle$, so it is modular and Artinian, but it is not Noetherian.

A look at the radical of L in Example 5 gives us a clue of what goes wrong in this example. L has no coatoms, hence $\mathrm{r}(L) = 1$. This never happens in the lattice of ideals of a ring (with 1), where maximal proper ideals are always guaranteed to exist, and therefore $\mathrm{r}(L) < 1$.

The following lemma illustrates one of the key features of modular lattices, which we can loosely describe as follows: "Behavior in the lattice can be moved inside intervals without much loss".

Lemma 6. *Let L be a modular lattice, $a \in L$, and $(x_i)_{i \in I}$ a chain in L. Let $y_i = a \wedge x_i$ and $z_i = a \vee x_i$. If the chain $(x_i)_{i \in I}$ is an infinite ascending (resp. descending) chain, then so is at least one of $(y_i)_{i \in I}$ and $(z_i)_{i \in I}$.*

Proof. Let's consider the ascending case. The descending case is dual. Suppose $(x_i)_{i\in I}$ is infinite ascending, but both $(a\wedge x_i)_{i\in I}$ and $(a\vee x_i)_{i\in I}$ become stationary at $u = a\wedge x_k = a\wedge x_{k+1} = \dots$ and $v = a\vee x_k = a\vee x_{k+1} = \dots$.
Using modularity we get

$$\begin{aligned} x_k &= x_k \vee u \\ &= x_k \vee (a\wedge x_{k+1}) \\ &= (x_k\vee a)\wedge x_{k+1} \\ &= v\wedge x_{k+1} \\ &= x_{k+1} \end{aligned}$$

contradicting the assumption about $(x_i)_{i\in I}$. □

Corollary 7. *Let L be a modular lattice, $a\in L$. L is Noetherian (resp. Artinian) iff $[0,a]$ and $[a,1]$ are Noetherian (resp. Artinian).*

Corollary 8. *Let L be a modular lattice, and $m\in L$ a coatom. If $(x_i)_{i\in I}$ is an infinite ascending chain in L, then so is $y_i = m\wedge x_i$.*

The usual proof of the HLT, first considers the case when the radical is 0, and then the general case. The argument in the first case is lattice theoretic, as we illustrate next. Note that there is no assumption about the lattice being algebraic.

Proposition 9. *Let L be a complete modular lattice. If L is Artinian and radical free, i.e. $\mathrm{r}(L) = 0$, then L is Noetherian.*

Proof. Being Artinian, $\mathrm{r}(L)$ must be expressible as the meet of finitely many maximal elements $m_1,\dots,m_k$. If we had an infinite ascending chain $(x_i)_{i\in I}$, repeated application of Corollary 8 yields an infinite ascending chain $(x_i\wedge m_1\wedge\cdots\wedge m_k)_{i\in I}$. But $x_i\wedge m_1\wedge\cdots\wedge m_k \leq m_1\wedge\cdots\wedge m_k = \mathrm{r}(L) = 0$. □

This proof shows that if something is going to go wrong about L being Noetherian, it will show up below the radical. So, we look at the interval $[0,\mathrm{r}(L)]$, and the radical of this lattice. This gives rise to the radical series.

Definition 10. Let L be a complete lattice. We define the *Loewy radical series* of L as follows:

- $r_0(L) = 1$,
- for any ordinal σ, $r_{\sigma+1}(L) = \mathrm{r}([0, r_\sigma(L)])$,
- for a limit ordinal σ, $r_\sigma(L) = \bigwedge_{\alpha<\sigma} r_\alpha(L)$.

The smallest ordinal σ such that $r_{\sigma+1}(L) = r_\sigma(L)$ is called the *Loewy radical length* of L, and $r_\sigma(L)$ is called the ***hyper-radical*** of L. It is denoted by $r_\infty(L)$. We say that L is ***hyper-radical free*** if $r_\infty(L) = 0$.

Being hyper-radical free is precisely the extra condition needed to extend the HLT to complete modular lattices.

Theorem 11. *Let L be a complete modular lattice. If L is Artinian and hyper-radical free, then L is Noetherian.*

Proof. Being Artinian, L must have finite Loewy radical length. Therefore, $r_n(L) = 0$ for some $n \in \mathbb{N}$. For $i = 1, \ldots, n$, the interval $[r_i(L), r_{i-1}(L)]$, is modular; it is Artinian by Corollary 7; it is radical free by construction. By Proposition 9 it is Noetherian. By Corollary 7, L is Noetherian. □

The Loewy radical series is the dual construction of the *Loewy (socle) series*, see [4]. The dual of radical-free is semiatomic, i.e. when the socle is equal to 1. The dual of the hyper-radical we call the *hyper-socle.* A lattice is *hyper-semiatomic* if the hyper-socle is equal to 1. By duality we get the following theorem:

Theorem 11'. *Let L be a complete modular lattice. If L is Noetherian and hyper-semiatomic, then L is Artinian.*

After extending the HLT to hyper-radical free modular lattices, a number of questions arise. Is the "hyper-radical free" hypothesis necessary? Is it vacuous? We answer these questions with an example and a proposition. But first a lemma.

Lemma 12. *Let R be a ring, N a left R-module, and J the (Jacobson) radical of R. Then $J \cdot N \leq \mathrm{r}(N) \leq N$*

Proof. If M is a maximal submodule of N then N/M is simple and J is contained in $\mathrm{Ann}(N/M)$. In other words, $J \cdot N \leq M$. It follows that $J \cdot N \leq \mathrm{r}(N)$. □

Example 13. The hyper-radical free hypothesis in Theorem 11 is not vacuous, not even for the ideal lattice of a ring. The ideal lattice $L = \mathrm{Idl}(R)$ of the ring R of germs of $C^\infty(\mathbb{R})$ functions at $x = 0$, is not hyper-radical free, and its Loewy radical length is ω. To see this, note that R is a local ring with maximal ideal

$$M = \{f \in R | f(0) = 0\} = x \cdot R,$$

so this is the first radical of L, i.e. $J = M$. M has a single maximal submodule

$$M_2 = \{f \in M | f'(0) = 0\} = x^2 \cdot R,$$

so this is the second radical of L, and inductively,

$$r_n(L) = \{f \in R | f^{(i)}(0) = 0 \text{ for } i = 0, \ldots, n-1\} = x^n \cdot R.$$

Therefore,

$$r_\omega(L) = \bigcap_n (x^n \cdot R) = \{f \in R | f^{(i)}(0) = 0 \text{ for all } i\},$$

the ideal of germs of flat functions. Now, by Lemma 12 we have $J \cdot r_\omega(L) \leq r_{\omega+1}(L) \leq r_\omega(L)$. But $J \cdot r_\omega(L) = x \cdot R \cdot r_\omega(L) = x \cdot r_\omega(L) = r_\omega(L)$, so $r_{\omega+1}(L) = r_\omega(L)$. It is a well-known fact that there are non-zero flat functions, like $f(x) = \exp(-1/x^2)$. So, $r_\infty(L) = r_\omega(L) \neq 0$.

The hyper-radical free hypothesis in Theorem 11 is necessary.

Proposition 14. *If L is a complete Noetherian lattice, then it is hyper-radical free.*

Proof. If we had $r_\infty(L) > 0$, then the interval $[0, r_\infty(L)]$ would have no coatoms. Therefore, it must have an infinite ascending chain. □

This proposition, combined with the Hopkins-Levitzki Theorem, tells us that the lattice of left ideals of a left-Artinian ring is hyper-radical free. The proof given of Theorem 11, does not replace the standard proof of the HLT for rings, unless one finds a direct argument to show that the lattice of left ideals of a left-Artinian ring is hyper-radical free.

Example 13 was suggested by Mazur and Karagueuzian [9]. It would be nice to have a characterization of the class of rings which are hyper-radical free.

References

[1] Toma Albu. Certain artinian lattices are noetherian. applications to the relative hopkins-levitzki theorem. In F Oystaeyen, editor, *Methods in Ring Theory*, volume 129 of *NATO ASI Series*, pages 37–52, 1983.

[2] Toma Albu and Patrick F. Smith. Localization of modular lattices, krull dimension, and the hopkins-levitzki theorem (i). *Math. Proc. Camb. Phil. Soc.*, 120:87–101, 1996.

[3] Toma Albu and Patrick F. Smith. Localization of modular lattices, krull dimension, and the hopkins-levitzki theorem (ii). *Comm. in Algebra*, 25(4):1111–1128, 1997.

[4] Grigore Călugăreanu. *Lattice Concepts of Module Theory.* Kluwer, 2000.

[5] Pierre Grillet. *Algebra.* Wiley-Interscience, 1999.

[6] Charles Hopkins. Rings with minimal condition for left ideals. *Ann. of Math.*, 40:712–730, 1939.

[7] Nathan Jacobson. *Basic Algebra*, volume II. W.H.Freeman and Company, 1980.

[8] Jakob Levitzki. On rings which satisfy the minimum condition for the right-hand ideals. *Compositio Math.*, 7:214–222, 1939.

[9] Marcin Mazur and Dikran Karagueuzian. Personal Communication.

Department of Mathematical Sciences
State University of New York at Binghamton
Binghamton, NY 13902-6000
fer@math.binghamton.edu